12. cvičení - řešení

Příklad 1 (a) $\int |x|\mathrm{d}x$.

Zřejmě $|x| = \begin{cases} x, & x > 0 \\ 0, & x = 0 \\ -x, & x < 0 \end{cases}$ je spojitá funkce.

$$x \in (-\infty, 0) \implies \int |x|\mathrm{d}x = \int -x\mathrm{d}x \overset{c}{=} -\frac{x^2}{2} =: F_1(x)$$
$$x \in (0, \infty) \implies \int |x|\mathrm{d}x = \int x\mathrm{d}x \overset{c}{=} \frac{x^2}{2} =: F_2(x)$$

Platí: $\lim_{x\to 0-} F_1(x) = 0 = \lim_{x\to 0+} F_2(x)$. Funkce F_1, F_2 lze tedy v bodě 0 spojitě dodefinovat hodnotou 0 - dostávám tak F_1 definovanou a spojitou na $(-\infty, 0]$ a F_2 definovanou a spojitou na $[0, \infty)$.

Máme tedy: $F_1' = f$ na $(-\infty, 0)$ a $F_2' = f$ na $(0, \infty)$ pro $f(x) = |x|$. Dokonce $F_1(0) = F_2(0) = 0$. Definujme tedy funkci

$$F = \begin{cases} F_1(x), & x \in (-\infty, 0] \\ F_2(x), & x \in (0, \infty) \end{cases}.$$

Pak $\int f = F$ na $\mathbb{R}$.

Příklad 1 (b) $\int |1-x| + |1+x|\mathrm{d}x$.

Platí: $f(x) := |1-x| + |1+x| = \begin{cases} -2x, & x < -1 \\ 2, & x \in [-1, 1] \\ 2x, & x > 1 \end{cases}$ je spojitá funkce.

$$x \in (-\infty, -1) \implies \int f(x)\mathrm{d}x = \int -2x\mathrm{d}x \overset{c}{=} -x^2 \implies F_1(x) := -x^2, x \in \mathbb{R}$$
$$x \in (-1, 1) \implies \int f(x)\mathrm{d}x = \int 2\mathrm{d}x \overset{c}{=} 2x \implies F_2(x) := 2x, x \in \mathbb{R}$$
$$x \in (1, \infty) \implies \int f(x)\mathrm{d}x = \int 2x\mathrm{d}x \overset{c}{=} x^2 \implies: F_3(x) := x^2, x \in \mathbb{R}$$

Na intervalech $(-\infty, -1), (-1, 1), (1, \infty)$ máme k f spojité primitivní funcke po řadě F_1, F_2, F_3. Nyní je stačí jen slepit - tj. přičíst k nim konstanty tak, aby v krajních bodech intervalu na sebe pěkně navazovaly a my tak dostali spojitou funkci F.

$$-1 : \lim_{x\to -1-} F_1(x) = \lim_{x\to -1-} -x^2 = -1$$
$$\lim_{x\to -1+} F_2(x) = \lim_{x\to -1+} 2x = -2$$
$$1 : \lim_{x\to 1-} F_2(x) = \lim_{x\to 1-} 2x = 2$$
$$\lim_{x\to 1+} F_3(x) = \lim_{x\to 1+} x^2 = 1$$

Zřejmě $F_1(-1) - 1 = F_2(-1)$ a $F_2(1) = F_3(1) - 1$. Lepíme tedy funkce $F_1 - 1, F_2, F_3 - 1$.

$$F(x) := \begin{cases} F_1(x) - 1 = -x^2 - 1, & x \in (-\infty, -1] \\ F_2(x) = 2x, & x \in (-1, 1) \\ F_3(x) - 1) = x^2 - 1, & x \in [1, \infty) \end{cases}$$

Poznámka: mohli jsme volit i jiné lepení (celou funkci F lze libovolně posunout za pomoci aditivní konstanty).

Příklad 1 (c) $\int \max\{1, x^2\}\mathrm{d}x$.

Platí: $f(x) := \max\{1, x^2\} = \begin{cases} x^2, & x < -1 \vee x > 1 \\ 1, & x \in [-1, 1] \end{cases}$

$$x \in (-\infty, -1) \cup (1, \infty) \implies \int f(x)\mathrm{d}x = \int x^2 \mathrm{d}x \stackrel{c}{=} \frac{x^3}{3} \implies F_1(x) := \frac{x^3}{3}, x \in \mathbb{R}$$
$$x \in (-1, 1) \implies \int f(x)\mathrm{d}x = \int 1\mathrm{d}x \stackrel{c}{=} x \implies F_2(x) = x, x \in \mathbb{R}$$

Máme: $F_1' = f$ na $(-\infty, -1) \cup (1, \infty)$ a $F_2' = f$ na $(-1, 1)$ a F_1, F_2 jsou na $\mathbb{R}$ spojité. Zbývá nám tedy nalepit F_1 na F_2 v bodě -1 a F_2 na F_1 v bodě 1.

$$\begin{aligned} -1: \lim_{x \to -1-} F_1(x) &= \lim_{x \to -1-} \frac{x^3}{3} = \frac{-1}{3} \\ \lim_{x \to -1+} F_2(x) &= \lim_{x \to -1+} x = -1 \\ 1: \lim_{x \to 1-} F_2(x) &= \lim_{x \to 1-} x = 1 \\ \lim_{x \to 1+} F_3(x) &= \lim_{x \to 1+} \frac{x^3}{3} = \frac{1}{3} \end{aligned}$$

$$F(x) := \begin{cases} F_1(x) - \frac{2}{3} = \frac{x^3}{3} - \frac{2}{3}, & x \in (-\infty, -1] \\ F_2(x) = x, & x \in (-1, 1) \\ F_1(x) + \frac{2}{3} = \frac{x^3}{3} + \frac{2}{3}, & x \in [1, \infty) \end{cases}$$

Příklad 1 (d) $\int e^{-|x|}\mathrm{d}x$.

Platí $f(x) := e^{-|x|} = \begin{cases} e^x, & x \in (-\infty, 0] \\ e^{-x}, & x \in (0, \infty) \end{cases}$ je spojitá funkce na $\mathbb{R}$.

$$x \in (-\infty, 0) \implies \int f(x)\mathrm{d}x = \int e^x \mathrm{d}x = e^x \implies F_1(x) := e^x, x \in \mathbb{R}$$
$$x \in (-\infty, 0) \implies \int f(x)\mathrm{d}x = \int e^{-x} \mathrm{d}x = -e^{-x} \implies F_1(x) := -e^{-x}, x \in \mathbb{R}$$

$$\lim_{x \to 0-} F_1(x) = e^0 = 1 = \lim_{x \to 0+} F_2(x)$$

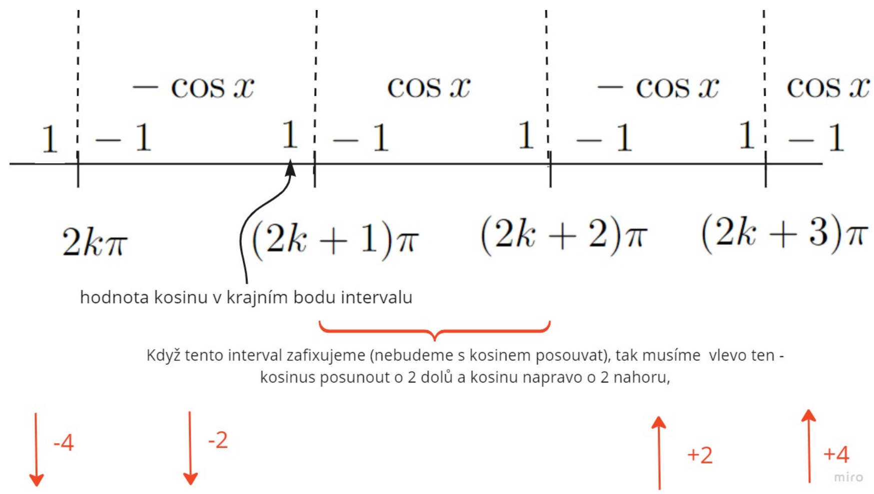
$$F(x) := \begin{cases} e^x, & x \in (-\infty, 0] \\ -e^{-x}, & x \in (0, \infty) \end{cases}$$

Příklad 1 (e) $\int |\sin x| \mathrm{d}x$.

$$f(x) := |\sin x| = \begin{cases} \sin x, & x \in [2k\pi, (2k+1)\pi], k \in \mathbb{Z} \\ -\sin x, & x \in ((2k+1)\pi, (2k+2)\pi), k \in \mathbb{Z} \end{cases}$$

$$x \in (2k\pi, (2k+1)\pi), k \in \mathbb{Z} \implies \int f(x)\mathrm{d}x = \int \sin x \mathrm{d}x \overset{c}{=} -\cos x \implies F_1(x) := -\cos x$$

$$x \in ((2k+1)\pi, (2k+2)\pi), k \in \mathbb{Z} \implies \int f(x)\mathrm{d}x = \int -\sin x \mathrm{d}x \overset{c}{=} \cos x \implies F_2(x) := \cos x$$



$$F(x) := \begin{cases} -\cos x, & x \in [0, \pi) \\ (-1)^{k-1}\cos x + 2k, & x \in [k\pi, (k+1)\pi), k \in \mathbb{N} \\ (-1)^{-k-1}\cos x - 2k, & x \in [(-k\pi, (-k+1)\pi), k \in \mathbb{N} \end{cases}$$

Příklad 1 (f) $\int \frac{1}{1+\sin^2 x}\mathrm{d}x$.

Pro racionální funkci $R(x,y) = \frac{1}{1+x^2}$ platí $R(-\sin x, -\cos x) = R(\sin x, \cos x)$. Budeme tedy volit substituci $t = \tan x$ pro výpočet primitivní funkce na intervalech $\left(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi\right), k \in \mathbb{Z}$. získané primitivní funkce se pak pokusíme slepit v bodech $\frac{\pi}{2} + k\pi, k \in \mathbb{Z}$.

$$x \in \left(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi\right), k \in \mathbb{Z} \implies \int \frac{1}{1+\sin^2 x}\mathrm{d}x = \left| t = \tan x, \mathrm{d}x = \frac{1}{1+t^2}\mathrm{d}t \right| =$$

$$= \int \frac{1}{1+\frac{t^2}{1+t^2}} \frac{1}{t^2+1}\mathrm{d}t = \int \frac{t^2+1}{1+2t^2}\frac{1}{t^2+1}\mathrm{d}t = \int \frac{1}{2t^2+1}\mathrm{d}t = \left| u = t\sqrt{2}, \mathrm{d}u = \sqrt{2}\mathrm{d}t \right| =$$

$$\overset{\text{lin.}}{=} \frac{1}{\sqrt{2}} \int \frac{1}{u^2+1}\mathrm{d}u \overset{c}{=} \frac{1}{\sqrt{2}} \arctan u = \frac{1}{\sqrt{2}} \arctan\left(t\sqrt{2}\right) = \frac{1}{\sqrt{2}} \arctan\left(\sqrt{2}\tan x\right)$$

Nyní zjistěme, jak se získaná primitivní funkce chová v bodech $\frac{\pi}{2} + k\pi, k \in \mathbb{Z}$.

$$\lim_{x \to \frac{\pi}{2} + k\pi -} \frac{1}{\sqrt{2}} \arctan\left(\sqrt{2}\tan x\right) \overset{\text{VOLSF}}{=} -\frac{\pi}{2\sqrt{2}}$$

$$\lim_{x \to \frac{\pi}{2} + k\pi +} \frac{1}{\sqrt{2}} \arctan\left(\sqrt{2}\tan x\right) \overset{\text{VOLSF}}{=} \frac{\pi}{2\sqrt{2}}$$

Podobně jako výše odvodíme vhodné posuny.

$$F(x)=\begin{cases}\frac{1}{\sqrt{2}}\arctan(\sqrt{2}\tan x)+k\frac{\pi}{\sqrt{2}}, & x\in\left(-\frac{\pi}{2}+k\pi,\frac{\pi}{2}+k\pi\right)\\(1+2k)\frac{\pi}{2\sqrt{2}}, & x=\frac{\pi}{2}+k\pi.\end{cases}$$

Příklad 1 (g) $\int\frac{1}{\sin x+\cos x+2}\mathrm{d}x$.

Pro $R(x,y):=\frac{1}{x+y+2}$ napletí ani jedno z následujícího: $R(-\sin x,\cos x)=R(\sin x,\cos x)$, $R(\sin x,-\cos x)=-R(\sin x,\cos x)$, $R(-\sin x,-\cos x)=R(\sin x,\cos x)$. Proto budeme volit substituci $t=\tan\frac{x}{2}$. Tu lze provést pouze na intervalech $(-\pi+k2\pi,\pi+k2\pi)\,,k\in\mathbb{Z}$. Budeme pak tedy muset v bodech $\pi+k2\pi$ lepit.

$$x\in(-\pi+k2\pi,\pi+k2\pi)\,,k\in\mathbb{Z}\implies\int\frac{1}{\sin x+\cos x+2}\mathrm{d}x=\left|t=\tan\frac{x}{2},\mathrm{d}x=\frac{2}{1+t^2}\mathrm{d}t\right|=$$

$$=\int\frac{1}{\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}+2}\cdot\frac{2}{1+t^2}\mathrm{d}t=\int\frac{1+t^2}{2t+1-t^2+2+2t^2}\cdot\frac{2}{1+t^2}\mathrm{d}t=$$

$$=\int\frac{2}{t^2+2t+3}\mathrm{d}t\overset{\text{lin.}}{=}2\int\frac{1}{(t+1)^2+2}\mathrm{d}t=\int\frac{1}{\left(\frac{t+1}{\sqrt{2}}\right)^2+1}\mathrm{d}t=\left|u=\frac{t+1}{\sqrt{2}},\mathrm{d}u=\frac{1}{\sqrt{2}}\mathrm{d}t\right|=$$

$$\overset{\text{lin.}}{=}\sqrt{2}\int\frac{1}{u^2+1}\mathrm{d}u\overset{\text{c}}{=}\sqrt{2}\arctan u=\sqrt{2}\arctan\frac{t+1}{\sqrt{2}}=\sqrt{2}\arctan\left(\frac{\tan\frac{x}{2}+1}{\sqrt{2}}\right)$$

K lepení nejdřív zjistěme, zda je možné získané primitivní funkce spojitě dodefinovat v krajních bodech intervalů.

$$\lim_{x\to\pi+k2\pi-}\sqrt{2}\arctan\left(\frac{\tan\frac{x}{2}+1}{\sqrt{2}}\right)=\frac{\pi\sqrt{2}}{2}=\frac{\pi}{\sqrt{2}}$$

$$\lim_{x\to\pi+k2\pi+}\sqrt{2}\arctan\left(\frac{\tan\frac{x}{2}+1}{\sqrt{2}}\right)=\frac{\pi\sqrt{2}}{2}=-\frac{\pi}{\sqrt{2}}$$

Pokud nebudeme se získanou primitivní funkcí hýbat na intervalu $(-\pi,\pi)$ (který odpovídá $k=0$), tak pak v intervalech napravo musíme v každém bodě lepení posunout funkci o $2\frac{\pi}{\sqrt{2}}=\pi\sqrt{2}$ nahoru. Naopak nalevo posouváme o $\pi\sqrt{2}$ dolů. Z toho dostáváme následující funkci.

$$F(x)=\begin{cases}\sqrt{2}\arctan\frac{\tan\frac{x}{2}+1}{\sqrt{2}}+k\pi\sqrt{2}, & x\in(-\pi+2k\pi,\pi+2k\pi)\\k\pi\sqrt{2}+\frac{\pi}{2}\sqrt{2}, & x=\pi+2k\pi.\end{cases}$$

Příklad 1 (h) $\int\frac{1}{3\cos^2x+\sin 2x+1}\mathrm{d}x$.

Pro $R(x,y)=\frac{1}{3y^2+2xy+1}$ platí následující: $R(-\sin x,-\cos x)=R(\sin x,\cos x)$. budeme tedy volit substituci $t=\tan x$, což lze rovést pouze na intervalech $\left(-\frac{\pi}{2}+k\pi,\frac{\pi}{2}+k\pi\right),k\in\mathbb{Z}$.

$$x \in (-\pi + k2\pi, \pi + k2\pi), k \in \mathbb{Z} \implies \int \frac{1}{3\cos^2 x + \sin 2x + 1} \mathrm{d}x = \int \frac{1}{3\cos^2 x + 2\sin x \cos x + 1} \mathrm{d}x =$$

$$= \left| t = \tan x, \mathrm{d}x = \frac{1}{1+t^2} \mathrm{d}t \right| = \int \frac{1}{3\frac{1}{1+t^2} + 2\frac{t}{1+t^2} + 1} \cdot \frac{1}{1+t^2} \mathrm{d}t = \int \frac{1+t^2}{3+2t+1+t^2} \cdot \frac{1}{1+t^2} \mathrm{d}t =$$

$$= \int \frac{1}{t^2+2t+4} \mathrm{d}t = \int \frac{1}{(t+1)^2+3} \mathrm{d}t \overset{\text{lin.}}{=} \frac{1}{3} \int \frac{1}{\left(\frac{t+1}{\sqrt{3}}\right)^2 + 1} \mathrm{d}t = \left| u = \frac{t+1}{\sqrt{3}}, \mathrm{d}u = \frac{1}{\sqrt{3}} \mathrm{d}t \right|$$

$$=\overset{\text{c}}{=} \frac{1}{\sqrt{3}} \arctan u = \frac{1}{\sqrt{3}} \arctan\left(\frac{1+\tan x}{\sqrt{3}}\right)$$

Ověřme, že lze provést lepení - tj., zda získaná primitivní funkce je spojitě definovatelná v krajních bodech intervalu.

$$\lim_{x \to \frac{\pi}{2}+k\pi-} \frac{1}{\sqrt{3}} \arctan\left(\frac{1+\tan x}{\sqrt{3}}\right) \overset{\text{VOLSF}}{=} \frac{\pi}{2\sqrt{3}}$$

$$\lim_{x \to \frac{\pi}{2}+k\pi+} \frac{1}{\sqrt{3}} \arctan\left(\frac{1+\tan x}{\sqrt{3}}\right) \overset{\text{VOLSF}}{=} -\frac{\pi}{2\sqrt{3}}$$

Podobnými úvahami jako dříve dostáváme následující lepení.

$$F(x) = \begin{cases} \frac{1}{\sqrt{3}} \arctan \frac{\tan x + 1}{\sqrt{3}} + k\frac{\pi}{\sqrt{3}}, & x \in \left(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi\right) \\ \frac{\pi}{2\sqrt{3}} + k\frac{\pi}{\sqrt{3}}, & x = \frac{\pi}{2} + k\pi. \end{cases}$$

Příklad 1 (i) $\int \frac{1}{6\cos^2 x + 4\sin x \cos x + \sin^2 x} \mathrm{d}x$.

Pro $R(x,y) = \frac{1}{6y^2+4xy+x^2}$ platí $R(-\sin x, -\cos x) = R(\sin x, \cos x)$. budeme tedy volit substituci $t = \tan x$, což lze rovést pouze na intervalech $\left(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi\right), k \in \mathbb{Z}$.

$$\int \frac{1}{6\cos^2 x + 4\sin x \cos x + \sin^2 x} \mathrm{d}x = \left| t = \tan x, \mathrm{d}x = \frac{1}{1+t^2} \mathrm{d}t \right| = \int \frac{1}{6\frac{1}{1+t^2} + 4\frac{t}{1+t^2} + \frac{t^2}{1+t^2}} \cdot \frac{1}{t^2+1} \mathrm{d}t =$$

$$= \int \frac{1+t^2}{6+4t+t^2} \cdot \frac{1}{t^2+1} \mathrm{d}t = \int \frac{1}{t^2+4t+6} \mathrm{d}t = \int \frac{1}{(t+2)^2+2} \mathrm{d}t \overset{\text{lin.}}{=} \frac{1}{2} \int \frac{1}{\left(\frac{t+2}{\sqrt{2}}\right)^2 + 1} =$$

$$= \left| u = \frac{t+1}{\sqrt{2}}, \mathrm{d}u = \frac{1}{\sqrt{2}} \mathrm{d}t \right| \overset{\text{c}}{=} \frac{1}{\sqrt{2}} \arctan u = \frac{1}{\sqrt{2}} \arctan \frac{\tan x + 1}{\sqrt{2}}$$

Ověřme, že lze provést lepení - tj., zda získaná primitivní funkce je spojitě definovatelná v krajních bodech intervalu.

$$\lim_{x \to \frac{\pi}{2}+k\pi-} \frac{1}{\sqrt{2}} \arctan \frac{\tan x + 1}{\sqrt{2}} \overset{\text{VOLSF}}{=} \frac{\pi}{2\sqrt{2}}$$

$$\lim_{x \to \frac{\pi}{2}+k\pi+} \frac{1}{\sqrt{2}} \arctan \frac{\tan x + 1}{\sqrt{2}} \overset{\text{VOLSF}}{=} -\frac{\pi}{2\sqrt{2}}$$

Podobnými úvahami jako dříve dostáváme následující lepení.

$$F(x) = \begin{cases} \frac{1}{\sqrt{2}} \arctan \frac{\tan x + 2}{\sqrt{2}} + k\frac{\pi}{\sqrt{2}}, & x \in \left(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi\right) \\ \frac{\pi}{2\sqrt{2}} + k\frac{\pi}{\sqrt{2}}, & x = \frac{\pi}{2} + k\pi. \end{cases}$$

Příklad 2 (a) $\int_{-3}^{7} x^3 - 2x + 1\mathrm{d}x$

$$\int_{-3}^{7} x^3 - 2x + 1\mathrm{d}x = \left[\frac{x^4}{4} - x^2 + x\right]_{-3}^{7} = \left(\frac{7^4}{4} - 49 + 7\right) - \left(\frac{81}{4} - 9 - 3\right) = 580 - 40 + 10 = 550$$

Příklad 2 (b) $\int_0^3 |1 - x|\mathrm{d}x$
Použijeme aditivitu integrálu

$$\int_0^3 |1-x|\mathrm{d}x = \int_0^1 1 - x\mathrm{d}x + \int_1^3 x - 1\mathrm{d}x = \left[x - \frac{x^2}{2}\right]_0^1 + \left[\frac{x^2}{2} - x\right]_1^3 =$$
$$= \left(1 - \frac{1}{2}\right) - \left(0 - \frac{0^2}{2}\right) + \left(\frac{9}{2} - 3\right) - \left(\frac{1}{2} - 1\right) = \frac{1}{2} - 0 + \frac{3}{2} + \frac{1}{2} = \frac{5}{2}$$

Příklad 2 (c) $\int_0^{2\pi} 2\sin^2 x\mathrm{d}x$

$$\int_0^{2\pi} 2\sin^2 x\mathrm{d}x \overset{\text{lin.}}{=} 2\int_0^{2\pi} \sin x \cdot \sin x \overset{\text{PP}}{=} 2\left[-\sin x\cos x\right]_0^{2\pi} - 2\int_0^{2\pi} -\cos^2 x\mathrm{d}x =$$
$$= 2\left[-\sin x\cos x\right]_0^{2\pi} - 2\int_0^{2\pi} -(1 - \sin^2 x)\mathrm{d}x = 2\left[-\sin x\cos x\right]_0^{2\pi} - 2\int_0^{2\pi} \sin^2 x - 1\mathrm{d}x =$$
$$= 2\left[-\sin x\cos x\right]_0^{2\pi} - 2\int_0^{2\pi} \sin^2 x\mathrm{d}x + 2\left[x\right]_0^{2\pi} = 2\left[x - \sin x\cos x\right]_0^{2\pi} - \int_0^{2\pi} 2\sin^2 x\mathrm{d}x$$
$$\implies 2\int_0^{2\pi} 2\sin^2 x\mathrm{d}x = 2\left[x - \sin x\cos x\right]_0^{2\pi} \implies \int_0^{2\pi} 2\sin^2 x\mathrm{d}x = \left[x - \sin x\cos x\right]_0^{2\pi} =$$
$$= (2\pi - \sin(2\pi)\cos(2\pi)) - (0 - \sin 0\cos 0) = 2\pi$$

Příklad 2 (d) $\int_{\frac{1}{e}}^{e} |\log x|\mathrm{d}x$

Použijeme aditivitu. Navíc platí: $\int \log x\mathrm{d}x = x\log x - x$. (známo z předchozích cvičení, lze spočíst pomocí per partes).

$$\int_{\frac{1}{e}}^{e} |\log x|\mathrm{d}x = \int_{\frac{1}{e}}^{1} -\log x\mathrm{d}x + \int_1^e \log x\mathrm{d}x = \left[x - x\log x\right]_{\frac{1}{e}}^{1} + \left[x\log x - x\right]_1^e =$$
$$= (1 - 0) - \left(\frac{1}{e} - \frac{1}{e}\log\frac{1}{e}\right) + (e\log e - e) - (0 - 1) = 1 - \frac{1}{e} + \frac{1}{e}(\log 1 - \log e) + e - e + 1 =$$
$$= 1 - \frac{1}{e} - \frac{1}{e} + 1 = 2 - \frac{2}{e}$$

Příklad 2 (e) $\int_0^{\pi} x^2\cos^2 x\mathrm{d}x$ Výše jsme spočetli, že $\int \sin^2 x\mathrm{d}x = \frac{1}{2}(x - \sin x\cos x)$. Z toho plyne:

$$\int \cos^2 x\mathrm{d}x = \int 1 - \sin^2 x\mathrm{d}x = x - \int \sin^2 x = x - \frac{1}{2}(x - \sin x\cos x) = \frac{1}{2}(x + \sin x\cos x)$$

Získanou rovnost využijeme níže.

$$\int_0^\pi x^2\cos^2 x\mathrm{d}x \overset{\text{PP}}{=} \left[x^2\frac{1}{2}\left(x+\sin x\cos x\right)\right]_0^\pi - \int_0^\pi 2x\frac{1}{2}\left(x+\sin x\cos x\right)\mathrm{d}x =$$
$$= \left[x^2\frac{1}{2}\left(x+\sin x\cos x\right)\right]_0^\pi - \int_0^\pi x^2 + x\sin x\cos x\mathrm{d}x =$$
$$= \left[x^2\frac{1}{2}\left(x+\sin x\cos x\right) - \frac{x^3}{3}\right]_0^\pi - \int_0^\pi x\sin x\cos x\mathrm{d}x$$

Spočteme $\int x\sin x\cos x\mathrm{d}x$ pomocí per partes.

$$\int \sin x\cos x\mathrm{d}x \overset{\text{PP}}{=} -\cos^2 x - \int(-\cos x)(-\sin x)\mathrm{d}x \implies \int \sin x\cos x\mathrm{d}x \overset{\text{c}}{=} -\frac{\cos^2 x}{2}$$
$$\int x\sin x\cos x\mathrm{d}x \overset{\text{PP}}{=} -\frac{x\cos^2 x}{2} + \int\frac{\cos^2 x}{2}\mathrm{d}x \overset{\text{c}}{=} -\frac{x\cos^2 x}{2} + \frac{1}{4}\left(x+\sin x\cos x\right)$$

Dosadíme do předchozího výpočtu.

$$\left[x^2\frac{1}{2}\left(x+\sin x\cos x\right) - \frac{x^3}{3}\right]_0^\pi - \int_0^\pi x\sin x\cos x\mathrm{d}x =$$
$$= \left[\frac{x^3}{2} + \frac{x^2}{2}\sin x\cos x - \frac{x^3}{3} + \frac{x\cos^2 x}{2} - \frac{1}{4}x - \frac{\sin x\cos x}{4}\right]_0^\pi =$$
$$= \left[\frac{x^3}{6} + \frac{x^2}{2}\sin x\cos x + \frac{x\cos^2 x}{2} - \frac{1}{4}x - \frac{\sin x\cos x}{4}\right]_0^\pi = \left(\frac{\pi^3}{6}+\frac{\pi}{2}-\frac{\pi}{4}\right) - 0 = \frac{\pi^3}{6}+\frac{\pi}{4} = \frac{2\pi^3+3\pi}{12}$$

Příklad 2 (f) $\int_0^{\sqrt{3}} x\arctan x\mathrm{d}x$

$$\int_0^{\sqrt{3}} x\arctan x\mathrm{d}x \overset{\text{PP}}{=} \left[\frac{x^2}{2}\arctan x\right]_0^{\sqrt{3}} - \frac{1}{2}\int_0^{\sqrt{3}}\frac{x^2+1-1}{1+x^2}\mathrm{d}x =$$
$$= \left[\frac{x^2}{2}\arctan x\right]_0^{\sqrt{3}} - \frac{1}{2}\int_0^{\sqrt{3}} 1\mathrm{d}x + \frac{1}{2}\int_0^{\sqrt{3}}\frac{1}{x^2+1}\mathrm{d}x = \left[\frac{x^2}{2}\arctan x - \frac{x}{2} + \frac{1}{2}\arctan x\right]_0^{\sqrt{3}} =$$
$$= \left(\frac{3}{2}\arctan\sqrt{3} - \frac{\sqrt{3}}{2} + \frac{1}{2}\arctan\sqrt{3}\right) - \left(0\cdot\arctan 0 - 0 + \frac{1}{2}\arctan 0\right) = \frac{3}{2}\cdot\frac{\pi}{3} - \frac{\sqrt{3}}{2} + \frac{1}{2}\cdot\frac{\pi}{3} - 0 =$$
$$= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$

Příklad 2 (g) $\int_0^{\frac{\pi}{2}} e^x\sin x\mathrm{d}x$

$$\int e^x\sin x\mathrm{d}x \overset{\text{PP}}{=} -e^x\cos x + \int e^x\cos x\mathrm{d}x \overset{\text{PP}}{==} -e^x\cos x + e^x\sin x - \int e^x\sin x\mathrm{d}x$$
$$\implies \int e^x\sin x\mathrm{d}x \overset{\text{c}}{=} \frac{1}{2}e^x\left(\sin x - \cos x\right)$$

$$\int_0^{\frac{\pi}{2}} e^x \sin x \mathrm{d}x = \left[\frac{1}{2}e^x\left(\sin x - \cos x\right)\right]_0^{\frac{\pi}{2}} = \frac{1}{2}e^{\frac{\pi}{2}}\left(1-0\right) - \frac{1}{2}e^0(0-1) = \frac{1}{2}e^{\frac{\pi}{2}} + \frac{1}{2}$$

Příklad 2 (h) $\int_0^{\log 4} xe^{-x}\mathrm{d}x$

$$\int xe^{-x}\mathrm{d}x \overset{\text{PP}}{=} -xe^{-x} + \int e^{-x}\mathrm{d}x \overset{\text{c}}{=} -xe^{-x} - e^{-x}$$

$$\int_0^{\log 4} xe^{-x}\mathrm{d}x = \left[-xe^{-x} - e^{-x}\right]_0^{\log 4} = -\frac{\log 4 - 1}{e^{\log 4}} - (0 - e^0) = -\frac{1}{4}(2\log 2) - \frac{1}{4} + 1 = -\frac{\log 2}{2} + \frac{3}{4}$$

Příklad 2 (i) $\int_0^{10\pi} \sqrt{1-\cos 2x}\mathrm{d}x$

$$\int \sqrt{1-\cos 2x}\mathrm{d}x = \int \sqrt{1-\left(\cos^2 x - \sin^2 x\right)}\mathrm{d}x = \int \sqrt{1-\cos^2 x + \sin^2 x}\mathrm{d}x = \int \sqrt{2\sin^2 x}\mathrm{d}x$$

Z aditivity integrálu a periodicity funkce $\sin x$ dostáváme: $\int_0^{10\pi} \sqrt{1-\cos 2x}\mathrm{d}x = 5\int_0^{2\pi} \sqrt{1-\cos 2x}\mathrm{d}x$. Dále:

$$\begin{aligned}
&\int_0^{2\pi} \sqrt{1-\cos 2x}\mathrm{d}x = \int_0^{2\pi} \sqrt{2\sin^2 x}\mathrm{d}x = \int_0^{\pi} \sqrt{2\sin^2 x}\mathrm{d}x + \int_{\pi}^{2\pi} \sqrt{2\sin^2 x}\mathrm{d}x = \\
&= \int_0^{\pi} \sqrt{2}\sin x\mathrm{d}x - \int_{\pi}^{2\pi} \sqrt{2}\sin x\mathrm{d}x = \left[-\sqrt{2}\cos x\right]_0^{\pi} + \left[\sqrt{2}\cos x\right]_{\pi}^{2\pi} = \\
&= \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} = 4\sqrt{2}
\end{aligned}$$

Dostáváme tedy, že $\int_0^{10\pi} \sqrt{1-\cos 2x}\mathrm{d}x = 5\int_0^{2\pi} \sqrt{1-\cos 2x}\mathrm{d}x = 5\cdot 4\sqrt{2} = 20\sqrt{2}$.

Příklad 2 (j) $\int_0^1 x^{15}\sqrt{1+3x^8}\mathrm{d}x$

$$\begin{aligned}
&\int_0^1 x^{15}\sqrt{1+3x^8}\mathrm{d}x = \left|y = 1+3x^8, \mathrm{d}y = 24x^7\mathrm{d}x, 0 \to 1+3\cdot 0^8 = 1, 1 \to 1+3\cdot 1^8 = 4\right| = \\
&= \int_1^4 \frac{y-1}{3\cdot 24}\sqrt{y}\mathrm{d}y \overset{\text{lin.}}{=} \frac{1}{72}\int_1^4 y^{\frac{3}{2}} - y^{\frac{1}{2}}\mathrm{d}y = \frac{1}{72}\left[\frac{2}{5}y^{\frac{5}{2}} - \frac{2}{3}y^{\frac{3}{2}}\right]_1^4 = \\
&= \frac{1}{72}\left(\frac{2\sqrt{4^5}}{5} - \frac{2\sqrt{4^3}}{3}\right) - \frac{1}{72}\left(\frac{2}{5} - \frac{2}{3}\right) = \frac{29}{270}
\end{aligned}$$

Příklad 2 (k) $\int_0^{\frac{\pi}{2}} \frac{1}{2\sin^2 x + 3\cos^2 x}\mathrm{d}x$

$$\int_0^{\frac{\pi}{2}} \frac{1}{2\sin^2 x + 3\cos^2 x}\mathrm{d}x = \left| t = \tan x, 0 \to 0, \frac{\pi}{2} \to \infty \right| = \int_0^{\infty} \frac{1}{2\frac{t^2}{t^2+1} + 3\frac{1}{1+t^2}} \cdot \frac{1}{1+t^2}\mathrm{d}t =$$
$$= \int_0^{\infty} \frac{1+t^2}{2t^2+3} \cdot \frac{1}{1+t^2}\mathrm{d}t = \int_0^{\infty} \frac{1}{2t^2+3}\mathrm{d}t \overset{\text{lin.}}{=} \frac{1}{3}\int_0^{\infty} \frac{1}{1+\left(\frac{t\sqrt{2}}{\sqrt{3}}\right)^2}\mathrm{d}t =$$
$$= \left| u = t\sqrt{\frac{2}{3}}, \mathrm{d}u = \sqrt{\frac{2}{3}}\mathrm{d}t, 0 \to 0, \infty \to \infty \right| \overset{\text{lin.}}{=} \frac{\sqrt{3}}{3\sqrt{2}}\int_0^{\infty} \frac{1}{1+u^2}\mathrm{d}u =$$
$$= \frac{1}{\sqrt{6}}\left[\arctan u\right]_0^{\infty} = \frac{1}{\sqrt{6}}\left(\lim_{u\to\infty} \arctan u\right) - \frac{1}{\sqrt{6}} \cdot \arctan 0 = \frac{\pi}{2\sqrt{6}}$$

Příklad 2 (l) $\int_0^{2\pi} \frac{1}{(2+\cos x)(3+\cos x)}\mathrm{d}x$

Provedeme substituci $t = \tan\frac{x}{2}$, kterou je možné provést na intervalech $(-\pi + 2k\pi, \pi + 2k\pi), k \in \mathbb{Z}$. Proto interval $(0, 2\pi)$, na kterém integrujeme, rozdělíme na $(0,\pi), (\pi, 2\pi)$ (díky aditivitě integrálu).

$$\int_0^{\pi} \frac{1}{(2+\cos x)(3+\cos x)}\mathrm{d}x = \left| t = \tan\frac{x}{2}, 0 \to 0, \pi \to \infty \right| = \int_0^{\infty} \frac{1}{\left(2+\frac{1-t^2}{1+t^2}\right)\left(3+\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2}{1+t^2}\mathrm{d}t =$$
$$= \int_0^{\infty} \frac{(1+t^2)^2}{(2+2t^2+1-t^2)(3+3t^2+1-t^2)} \cdot \frac{2}{1+t^2}\mathrm{d}t = \int_0^{\infty} \frac{2+2t^2}{(t^2+3)(2t^2+4)}\mathrm{d}t =$$
$$\overset{\text{lin.}}{=} \int_0^{\infty} \frac{2}{t^2+3} - \frac{1}{t^2+2}\mathrm{d}t = \left[\frac{2}{\sqrt{3}}\arctan\frac{t}{\sqrt{3}} - \frac{1}{\sqrt{2}}\arctan\frac{t}{\sqrt{2}}\right]_0^{\infty} =$$
$$= \left(\lim_{t\to\infty} \frac{2}{\sqrt{3}}\arctan\frac{t}{\sqrt{3}} - \frac{1}{\sqrt{2}}\arctan\frac{t}{\sqrt{2}}\right) - 0 \overset{\text{VOLSF}}{=} \frac{2}{\sqrt{3}} \cdot \frac{\pi}{2} - \frac{1}{\sqrt{2}} \cdot \frac{\pi}{2} = \frac{\pi}{\sqrt{3}} - \frac{\pi}{\sqrt{8}}$$

$$\int_{\pi}^{2\pi} \frac{1}{(2+\cos x)(3+\cos x}\mathrm{d}x = \left| t = \tan\frac{x}{2}, \pi \to -\infty, 2\pi \to 0 \right| = \int_{-\infty}^{0} \frac{1}{\left(2+\frac{1-t^2}{1+t^2}\right)\left(3+\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2}{1+t^2}\mathrm{d}t =$$
$$\overset{\text{lin.}}{=} \int_{-\infty}^{0} \frac{2}{t^2+3} - \frac{1}{t^2+2}\mathrm{d}t = \left[\frac{2}{\sqrt{3}}\arctan\frac{t}{\sqrt{3}} - \frac{1}{\sqrt{2}}\arctan\frac{t}{\sqrt{2}}\right]_{-\infty}^{0} =$$
$$= 0 - \left(\lim_{t\to-\infty} \frac{2}{\sqrt{3}}\arctan\frac{t}{\sqrt{3}} - \frac{1}{\sqrt{2}}\arctan\frac{t}{\sqrt{2}}\right) = -\left(\frac{2}{\sqrt{3}} \cdot \frac{-\pi}{2} - \frac{1}{\sqrt{2}} \cdot \frac{-\pi}{2}\right) = -\left(-\frac{\pi}{\sqrt{3}} + \frac{\pi}{\sqrt{8}}\right) =$$
$$= \frac{\pi}{\sqrt{3}} - \frac{\pi}{\sqrt{8}}$$

$$\int_0^{2\pi} \frac{1}{(2+\cos x)(3+\cos x)}\mathrm{d}x = \int_0^{\pi} \frac{1}{(2+\cos x)(3+\cos x)}\mathrm{d}x + \int_{\pi}^{2\pi} \frac{1}{(2+\cos x)(3+\cos x)}\mathrm{d}x = \frac{2\pi}{\sqrt{3}} - \frac{\pi}{\sqrt{2}}$$

Příklad 3 (a) $\int \frac{\tan x}{6+11\cos x+6\cos^2 x+\cos^3 x}\mathrm{d}x$

$$\int \frac{\tan x}{6+11\cos x+6\cos^2 x+\cos^3 x}\mathrm{d}x = |t=\cos x, \mathrm{d}t = -\sin x \mathrm{d}x| = \int \frac{-\frac{1}{t}}{t^3+6t^2+11t+6}\mathrm{d}t =$$
$$= \int \frac{-1}{t(t+1)(t+2)(t+3)}\mathrm{d}t = \int \frac{\frac{1}{2}}{t+1} - \frac{\frac{1}{2}}{t+2} + \frac{\frac{1}{6}}{t+3} - \frac{\frac{1}{6}}{t}\mathrm{d}t =$$
$$\overset{c}{=} \frac{1}{2}\log|t+1| - \frac{1}{2}\log|t+2| + \frac{1}{6}\log|t+3| - \frac{1}{6}\log|t| =$$
$$= \frac{1}{2}\log|\cos x+1| - \frac{1}{2}\log|\cos x+2| + \frac{1}{6}\log|\cos x+3| - \frac{1}{6}\log|\cos x|$$

Příklad 3(m) $\int_2^\infty \frac{1}{x}\cdot\frac{1}{\log^3 x+2\log x}\mathrm{d}x$

$$\int_2^\infty \frac{1}{x}\cdot\frac{1}{\log^3 x+2\log x}\mathrm{d}x = \left| y=\log x, \mathrm{d}y = \frac{1}{x}\mathrm{d}x, 2\to\log 2, \infty\to\infty \right| = \int_{\log 2}^\infty \frac{1}{y^3+2y}\mathrm{d}y$$

Parciální zlomky:

$$\frac{1}{y^3+2y} = \frac{1}{y(y^2+2)} = \frac{A}{y} + \frac{By+C}{y^2+2}$$
$$1 = Ay^2+2A+By^2+Cy = y^2(A+C)+Dy+2A \implies \qquad 0 = A+C$$
$$0 = D$$
$$1 = 2A$$

Tedy $\frac{1}{y^3+2y} = \frac{\frac{1}{2}}{y} + \frac{-\frac{1}{2}y}{y^2+2}$.

$$\int \frac{\frac{1}{2}}{y} + \frac{-\frac{1}{2}y}{y^2+2}\mathrm{d}y \overset{\text{lin.}}{=} \frac{1}{2}\int\frac{1}{y}\mathrm{d}y - \frac{1}{4}\int\frac{2y}{y^2+2}\mathrm{d}y = \left|z=y^2+2, \mathrm{d}z = 2y\mathrm{d}y\right| = \frac{1}{2}\log|y| - \frac{1}{4}\int\frac{1}{z}\mathrm{d}z =$$
$$\overset{c}{=} \frac{1}{2}\log|y| - \frac{1}{4}\log(y^2+2)$$

$$\int_{\log 2}^\infty \frac{1}{y^3+2y}\mathrm{d}y = \left[\frac{1}{2}\log|y| - \frac{1}{4}\log(y^2+2)\right]_{\log 2}^\infty = \left[\log\frac{\sqrt{y}}{\sqrt[4]{y^2+2}}\right]_{\log 2}^\infty =$$
$$= \lim_{y\to\infty}\log\frac{\sqrt{y}}{\sqrt[4]{y^2+2}} - \lim_{y\to\log 2-}\log\frac{\sqrt{y}}{\sqrt[4]{y^2+2}} = 0 - \lim_{y\to\log 2-}\frac{1}{4}\log\frac{y^2}{y^2+2} = \frac{1}{4}\log\frac{\log^2 2}{\log^2 2+2}$$